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Marginal deformations of $\mathcal{N}$=4 SYM and Penrose limits with continuous spectrum

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Abstract: We study the Penrose limit about a null geodesic with 3 equal angular momenta in the recently obtained type IIB solution dual to an exactly marginal γ-deformation of $\mathcal{N} = 4$ SYM. The resulting background has non-trivial NS 3-form flux as well as RR 5- and 3-form fluxes. We quantise the light-cone Green-Schwarz action and show that it exhibits a continuum spectrum. We show that this is related to the dynamics of a charged particle moving in a Landau plane with an extra interaction induced by the deformation. We interpret the results in the dual $\mathcal{N} = 1$ SCFT.

Keywords: AdS-CFT and dS-CFT Correspondence, Penrose limit and pp-wave background.

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1. Introduction

Field theories with conformal symmetry often correspond to isolated points in the space of couplings. The reason is that conformal symmetry implies the vanishing of all β-functions, which imposes one relation for each coupling. As is well known [1], supersymmetric field theories can circumvent this argument because the β-functions can be expressed in terms of the anomalous dimensions of the fundamental chiral fields. If the theory has less anomalous dimensions than couplings, then not all the relations that follow from the vanishing of the β-functions are independent, leading to a continuous family of conformal theories. If, in addition, these field theories have a supergravity dual, the AdS/CFT correspondence [2 – 4] implies the existence of continuous families of supergravity solutions with AdS factors.

Lunin and Maldacena [5] have recently provided a method to construct such families for field theories with, at least, $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ global symmetry. The essence of the method is to use an $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(3,\mathbb{R})$ transformation of the full $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ duality group of IIB supergravity compactified along the corresponding $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ torus. Note that exactly the same method had been used a few years ago to generate the holographic duals of noncommutative field theories [6, 7], the only difference being that the torus is now transverse to the branes.

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In this paper we will concentrate on the exactly marginal deformations of the 4d $\mathcal{N}$=4 SU(N) SYM which are generated by this procedure. The mentioned resemblance with noncommutative deformations led the authors of [5] to propose that the resulting field theories are such that the standard products among the fields are replaced by

$$\Phi_1\,\Phi_2 \;\to\; e^{i\pi\gamma(Q^1_{\Phi_1}Q^2_{\Phi_2}-Q^1_{\Phi_2}Q^2_{\Phi_1})}\Phi_1\Phi_2\,, \tag{1.1}$$

where $(Q^1_{\Phi_i}, Q^2_{\Phi_i})$ are the charges of Φ_i under the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ action, and γ is the deformation parameter.[1,2]

Let us stress that this modification does not lead to a spacetime noncommutative theory, it just introduces some phases in the operators of the theory; for example, the $\mathcal{N}$=4 superpotential is modified

$$\mathrm{tr}\,(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \;\to\; \mathrm{tr}\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right), \tag{1.2}$$

where the three Φ_i are the complex $\mathcal{N} = 1$ chiral superfields, each with unit charge with respect to a generator J_{ϕ_i} of the Cartan subalgebra of the SO(6) R-symmetry. This is precisely one of the deformations of $\mathcal{N} = 4$ that had been proven to be exactly marginal purely from field theoretical arguments [1]. In particular, it breaks $\mathcal{N} = 4$ to $\mathcal{N}$=1 while preserving only the Cartan subgroup of SO(6).

Among the many new features of the deformed theory, we will be concerned with the fact that the set of chiral operators is modified. Recall that in the undeformed theory there exist 1/2-BPS operators with any set of integer values of $(J_{\phi_1}, J_{\phi_1}, J_{\phi_1})$, given simply by

$$\mathcal{O} = \mathrm{str}\left(\Phi_1^{J_{\phi_1}}\Phi_2^{J_{\phi_2}}\Phi_3^{J_{\phi_3}}\right), \tag{1.3}$$

where 'str' stands for 'symmetrized trace'. Once the deformation is turned on, these operators are multiplied as in (1.1), introducing a complicated set of relative phases among the various terms in the sum. The result is that for generic values of γ the spectrum of single-trace 1/2-BPS operators reduces to those carrying either of the following charges [5, 15, 16]

$$(J_{\phi_1}, J_{\phi_2}, J_{\phi_3}) = (n,0,0)\,, \quad (0,n,0)\,, \quad (0,0,n)\,, \quad (n,n,n)\,, \qquad n \in \mathbb{Z}\,. \tag{1.4}$$

The supergravity counterpart of this fact is as follows. In the original $\mathcal{N} = 4$, the dual background is $AdS_5 \times S^5$. All null geodesics lying inside the S^5 are SO(6)-related, so we can start with one carrying only J_{ϕ_1} charge, apply SO(6) rotations and generate others where the three J_{ϕ_i} are non-zero. In the background dual to the deformed $\mathcal{N} = 1$ SCFT, the metric on the S^5 has been deformed so that we only have a $\mathrm{U}(1)^3$ isometry at our disposal. Thus geodesics carrying charges as in (1.4) are isometrically inequivalent. Indeed, the background preserves an extra $\mathbb{Z}_3$ symmetry which permutes the first three states of (1.4) and leaves the fourth one invariant. As a result, there are essentially two inequivalent Penrose limits about BPS geodesics in the deformed geometry.

[1] In general, γ can be complex, in which case the letter β is often used instead of γ. We will only consider the case when γ is *real*, and consequently use the name γ-deformation.

[2] See also [8 − 12] for recent work on this theory, and [13, 14] for extensions to superconformal field theories in three dimensions.

1.1 Results and spectrum

In this paper we will study how some properties of the deformed $\mathcal{N}=1$ SCFT are realised in the string theory dual. With this aim in mind, we will take the Penrose limit about the (n,n,n) geodesic.[3] We first show that the resulting background preserves 20 of the 32 supersymmetries. We then show that, despite the fact that such background (see eq. (2.9)) has three different types of fluxes turned on, namely the NS 3-form and the RR 5- and 3-forms, it turns out that the Green-Schwarz action is quadratic in the light-cone gauge, and therefore exactly solvable.

We will show that the Penrose limit focuses on operators with

$$\Delta, J_{\phi_1}, J_{\phi_2}, J_{\phi_3} \sim \mathcal{O}(R^2)\,, \qquad \Delta - (J_{\phi_1}+J_{\phi_2}+J_{\phi_3}) = \mathcal{O}(1)\,, \tag{1.5}$$

and forces us to scale γ such that

$$\hat{\gamma} \equiv \frac{\gamma}{\sqrt{3}}\frac{R^2}{l_s^2} = \text{fixed}\,, \tag{1.6}$$

where $R^4/l_s^4 \sim g_s N \to \infty$ in the limit. In particular, (1.6) requires $\gamma \ll 1$, ensuring that string theory in the resulting background is not $\mathrm{SL}(2,\mathbb{Z})$ equivalent to string theory in the maximally supersymmetric pp-wave.

Before writing down the string theory spectrum in the Penrose limit of the deformed background, let us first discuss what the results would be in the equivalent Penrose limit (focusing on states with quantum numbers as in (1.5)) of $AdS_5 \times S^5$. When $\gamma = 0$ there are a huge number of operators scaling as required by (1.5) and with $\Delta - (J_{\phi_1} + J_{\phi_2} + J_{\phi_3}) = 0$. These are obtained from the operator (1.3) by removing Φ_i and inserting one Φ_j, with $i \neq j$. We will refer to this as *exchanges* of $\Phi_i \leftrightarrow \Phi_j$. For large J's, this procedure generates infinitely many new states. The quantization of the string σ-model in corresponding Penrose limit of reflects this fact by producing a ground state with an *infinite discrete* degeneracy. Some of the particle-like modes (those which are homogeneous along the string) have dynamics corresponding to that of a particle in a plane threatened by a constant magnetic field, a Landau problem, which explains the ground state infinite degeneracy [17].

It is interesting to see how string theory knows about the fact that, as soon as $\gamma \neq 0$, none of the previous exchanges produces BPS-protected operators any longer, as we read from the series (1.4). We would therefore expect the string theory ground state to be unique, and we expect that all mentioned exchanges lead to operators with $\Delta - J > 0$. More explicitly, we expect the latter operators to have $\Delta - (J_{\phi_1} + J_{\phi_2} + J_{\phi_3}) = f(\gamma, J_{\phi_i})$, with f a positive function that vanishes if $\gamma = 0$ or if all J_{ϕ_i} are equal; it should therefore be proportional to γ and depend only on the differences between the J_{ϕ_i}.

At the level of the string zero modes, we find that the turning on of γ introduces an extra interaction in the Landau problem. This interaction is an attractive quadratic

[3] We refer the reader to [17] for a study of string theory in the Penrose limit about the $(n,0,0)$ geodesics, noting that these authors remarkably discovered the corresponding pp-wave geometry even before [5] appeared.

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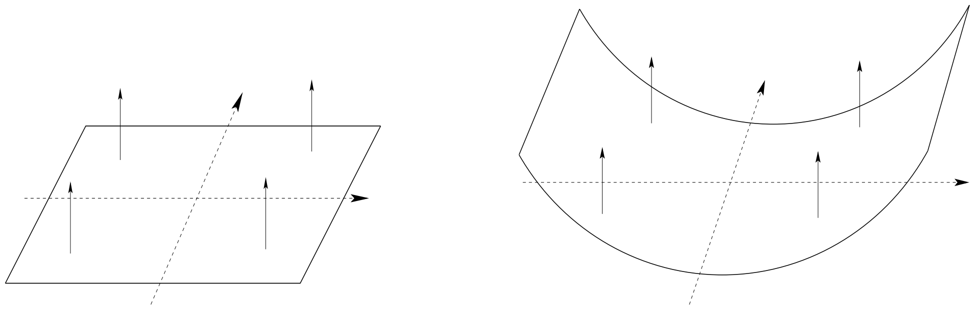

Figure 1: The physics as seen by the particle excitations of the string. Vertical arrows represent the magnetic field. Left: a Landau problem. Right: the modified Landau problem when $\gamma \neq 0$.

	level	number of modes	$\Delta - (J_{\phi_1} + J_{\phi_2} + J_{\phi_3})$
four pp-wave directions	$n = 0$	4	1
(origin: AdS)	$n \neq 0$	4 left-mov + 4 right-mov	$\sqrt{1 + \frac{n^2}{(\mu\alpha' p_v)^2}}$
four modified	$n = 0$	2	continuous
Landau directions	$n = 0$	2	2
(origin: S^5)	$n \neq 0$	4 left-mov + 4 right-mov	$\pm 1 + \sqrt{1 + \frac{n^2}{(\mu\alpha' p_v)^2}}$

Table 1: Spectrum of the resulting pp-wave.

potential along only one of the axis of the plane. We can visualize the problem as imagining that the plane is bent as in figure 1. Note that the asymptotics of the potential are radically changed no matter how small γ is. We find that this leads to a very different behavior of the system: the infinite discrete degeneracy is lifted, leaving behind a unique ground state and a continuous free-particle spectrum. Explicitly, we find the light-cone energy of these modes is

$$E_{l.c.} = \text{const.} \times \gamma^2 \times p^2 \,, \tag{1.7}$$

where p is the momentum along the undeformed direction of the Landau plane in figure 1. We will have more to say about the field theory interpretation of these results in section 5.

Let us finally gather here the results concerning the spectrum of the string, which is derived in section 4. Note that the $\mathrm{SL}(2,\mathbb{R})$ deformation leaves the AdS_5 part intact; as a result, the Penrose limit produces 4 transverse directions which are identical to the maximally supersymmetric pp-wave limit of [18]. However, the S^5 part of the geometry is modified and leads to 4 'modified Landau directions' with more complicated interactions. We will show that the background preserves 20 supersymmetries, 4 of them being 'supernumerary' [19] and therefore linearly realised. So, for simplicity, we only include the complete bosonic spectrum in the table 1.

Note that despite the fact that the background (and therefore the whole GS action) depends in a very non-trivial way on the deformation parameter γ, most modes have energies which are independent of it. The factors ± 1 of the Landau frequencies relative to the standard pp-wave ones are easily explained [20, 17] by the twisting in the worldsheet hamiltonian that is induced when changing from coordinates adapted to focus on geodesics with charges $(n, 0, 0)$ to charges (n, n, n).

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The organization of the paper is as follows. In section 2 we discuss the BPS null geodesics in the γ-deformed background and perform the Penrose limit along one of them. Section 3 shows that the resulting background preserves 20 supersymmetries. Section 4 is devoted to the quantization of the bosonic and fermionic sectors of the GS action. Finally, we discuss the implications of our results in section 5.

2. The Penrose limit of the deformed background

In this section we comment on the different null geodesics that the γ-deformed background has and we perform the Penrose limit that focuses on one of them.

We start with the type IIB $AdS_5 \times S^5$ background, with metric, RR 5-form and dilaton given by

$$\begin{aligned} ds^2 &= R^2 \left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\right) + R^2 \sum_{i=1}^{3} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right) \,, \\ F_5 &= 4R^4 e^{-\phi_0} \left(\omega_{AdS_5} + \omega_{S^5}\right) \,, \\ e^{\phi} &= e^{2\phi_0} \,, \end{aligned} \tag{2.1}$$

with $(\mu_1, \mu_2, \mu_3) = (\cos\alpha, \sin\alpha\cos\theta, \sin\alpha\sin\theta)$ and $R^4 = 4\pi e^{\phi_0} N l_s^4$. Out of the SO(6) isometry group of the round S^5, the coordinates chosen in (2.1) exhibit only a manifest $\mathrm{U}(1)^3$ subgroup which acts as shifts of the 3 angles ϕ_i. These will lead to conserved charges in the string worldsheet which we will name J_{ϕ_i}. In the dual gauge theory, each of these generators rotates the corresponding complex $\mathcal{N}=1$ scalar superfield Φ_i with unit charge.

To obtain the background dual to the β-deformed SCFT, we first need to select two U(1) symmetries, compactify along them to 8d, and then perform an $\mathrm{SL}(2,\mathbb{R})$ transformation. Lunin and Maldacena [5] chose a $\mathrm{U}(1)^2$ subgroup generated by a certain linear combination of the J_{ϕ_i} defined above. Namely, by defining three new angles $(\psi, \varphi_1, \varphi_2)$ via

$$\phi_1 = \psi + \varphi_1 + \varphi_2 \,, \qquad \phi_2 = \psi - \varphi_1 \,, \qquad \phi_3 = \psi - \varphi_2 \,, \tag{2.2}$$

the chosen $\mathrm{U}(1)^2$ acts as shifts of φ_1 and φ_2. For future reference, we will label the charges corresponding to shifts of the new 3 angles by $(J_\psi, J_{\varphi_1}, J_{\varphi_2})$, related to the previous ones by

$$J_\psi = J_{\phi_1} + J_{\phi_2} + J_{\phi_3} \,, \qquad J_{\varphi_1} = J_{\phi_1} - J_{\phi_2} \,, \qquad J_{\varphi_2} = J_{\phi_1} - J_{\phi_3} \,. \tag{2.3}$$

Note that all the scalars Φ_i have charge 1 under J_ψ and that the J_φ's measure deviations from $J_{\Phi_1} = J_{\Phi_2} = J_{\Phi_3}$. Being the generators of rotations along the torus, $(J_{\varphi_1}, J_{\varphi_2})$ are to be identified with the charges (Q^1, Q^2) of equation (1.1).

The resulting γ-deformed background has, in addition to the previous IIB fields, non-trivial RR and NS 3-forms. In order to analyze its null geodesics it will suffice by now to concentrate only on the resulting metric,

$$\begin{aligned} ds_\gamma^2 &= R^2 \left[ds_{AdS_5}^2 + \sum_{i=1}^{3} \left(d\mu_i^2 + G\mu_i^2 d\phi_i^2\right) + \frac{R^4}{l_s^4}\gamma^2 G \mu_1^2\mu_2^2\mu_3^2 \left(\sum_{i=1}^{3} d\phi_i\right)^2 \right] , \\ G^{-1} &\equiv 1 + \frac{R^4}{l_s^4}\gamma^2 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2) \,. \end{aligned} \tag{2.4}$$

All other fields are written in the appendix B. A quick look at the metric reveals that the S^5 has been deformed in such a way that the original SO(6) isometry has been broken to $\mathrm{U}(1)^3$, out of which only the U(1) generated by J_ψ remains an R-symmetry in the dual $\mathcal{N}=1$ SCFT. Let us label by S^5_γ the deformed sphere with metric as in (2.4) and note that because the deformation is a continuous one, its topology is still that of an S^5.

We now wish to perform the Penrose limit about a null geodesic lying inside the S^5_γ. In the original round S^5, we could use the full SO(6) symmetry to relate all such geodesics. In contrast, we now only have a $\mathrm{U}(1)^3$ group at our disposal and, as a result, Penrose limits about different geodesics may give rise to non diffeomorphic metrics. We do not aim at studying all such geodesics here, but we will rather concentrate on those that are BPS. At this point, we use information from the gauge theory [5, 15, 16], where we know that, for generic γ, the spectrum of single-trace BPS operators reduces to those carrying either of the following charges

$$(J_{\phi_1}, J_{\phi_2}, J_{\phi_3}) = (n,0,0)\,, \quad (0,n,0)\,, \quad (0,0,n)\,, \quad (n,n,n)\,, \qquad n\in\mathbb{Z}\,. \tag{2.5}$$

The corresponding null geodesics can be parametrized as follows: using τ as the worldline coordinate, and setting $t=\phi_1=\phi_2=\phi_3=\tau$, the charges in (2.5) are carried by massless particles along geodesics with

$$(\mu_1^2,\mu_2^2,\mu_3^2) = (1,0,0)\,, \quad (0,1,0)\,, \quad (0,0,1)\,, \quad \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \tag{2.6}$$

respectively. From the AdS_5 perspective they are just a point sitting at the origin in global coordinates.

The quantization of the string sigma model in the Penrose limit of the first three geodesics was studied by Niarchos and Prezas [17]. Here we will instead consider the fourth case. Such Penrose limit can be obtained by defining

$$\begin{aligned}
&t=u\,, \qquad \rho=\frac{r}{R}\,, \qquad \psi = u+\frac{v}{R^2}\,, \qquad \alpha=\alpha_0+\frac{y^1}{R}\,, \qquad \theta=\frac{\pi}{4}+\sqrt{\frac{3}{2}}\frac{y^3}{R}\,,\\
&\varphi_1 = \left(\frac{1}{2G}\right)^{1/2}\frac{-y^2+\sqrt{3}y^4}{R}\,, \qquad \varphi_2 = -\left(\frac{1}{2G}\right)^{1/2}\frac{y^2+\sqrt{3}y^4}{R}\,,
\end{aligned} \tag{2.7}$$

with $\alpha_0=\cos^{-1}1/\sqrt{3}$, and then sending $R\to\infty$ while keeping $\hat\gamma=\frac{\gamma}{\sqrt{3}}\frac{R^2}{l_s^2}$ fixed. In the limit, G tends to a constant,

$$G\to\frac{1}{1+\hat\gamma^2}\,, \tag{2.8}$$

and this is the value for G that will be used in the rest of the paper.

Using the expressions for the rest of the fields of the γ-deformed solution (B.1), we find that the resulting type IIB configuration is

$$\begin{aligned}
ds^2_{\mathrm{IIB}} &= 2dudv-\left[r^2+4G\hat\gamma^2\mu^2((y^1)^2+(y^3)^2)\right]du^2+4G^{1/2}\mu du(y^1dy^2+y^3dy^4)+d\vec{r}^{\,2}+d\vec{y}^{\,2}\,,\\
H_3 &= 2\hat\gamma G^{1/2}\mu\,(-dy^1\wedge dy^4+dy^3\wedge dy^2)\wedge du\,,\\
F_3 &= -\hat\gamma 4\mu\; du\wedge dy^1\wedge dy^3\,,\\
F_5 &= 4\,\mu\, du\,\wedge\,\left(dr^1\wedge dr^2\wedge dr^3\wedge dr^4+dy^1\wedge dy^2\wedge dy^3\wedge dy^4\right),\\
e^{2\phi} &= e^{2\phi_0}G\,,
\end{aligned} \tag{2.9}$$

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with $G^{-1} = 1 + \hat{\gamma}^2$. Here $\vec{r}$ and $\vec{y}$ parametrise two copies of $\mathbb{R}^4$, with r and y being the corresponding radial coordinates. We have introduced a mass parameter μ via $(u, v) \to (\mu u, v/\mu)$; this ensures that the metric tensor is dimensionless whereas all coordinates have dimensions of length. Note that in string theory, the light-cone momenta correspond to

$$\frac{p_u}{\mu} = -i\partial_u = \Delta - (J_{\Phi_1} + J_{\Phi_2} + J_{\Phi_3})\,, \qquad \mu p_v = -i\partial_v = R^{-2}(J_{\Phi_1} + J_{\Phi_2} + J_{\Phi_3})\,, \quad (2.10)$$

so that requiring that they be finite in the limit leads to (1.5).

Let us make a few remarks about the resulting background.

- Despite the fact that the dilaton is non-constant before taking the limit (see (B.1)), it becomes constant (although different from the $AdS_5 \times S^5$ case) after the limit.
- A crucial point is that the whole background admits a covariantly constant null Killing vector ∂_v. It can then be argued [21 − 24] that the fermionic GS action [25] is obtained from the flat one via the replacement of partial derivatives by the type IIB supercovariant ones (appearing e.g. in the variation of the gravitino).
- The metric exhibits a natural split between the four directions parametrized by $\vec{r}$ with AdS origin, and the four parametrized by $\vec{y}$ with S^5_γ origin; whereas the former remain identical to those of the maximally supersymmetric pp-wave, the latter will lead to more sophisticated physics.

We will find it more convenient to work in a slightly different gauge. After the coordinate transformation $v \to v - G^{1/2}(y^1y^2 + y^3y^4)$, the metric and NS 3-form can be written as

$$\begin{aligned} ds^2_{IIB} &= 2dudv - (r^2 + k_i y^i y^i)du^2 + 2f_{ij}y^i dy^j du + d\vec{r}^{\,2} + d\vec{y}^{\,2}\,, \\ H_3 &= h_{ij} dy^i \wedge dy^j \wedge du\,, \end{aligned} \quad (2.11)$$

with f_{ij} and h_{ij} antisymmetric and only nonzero components

$$f_{12} = f_{34} = \mu G^{1/2}\,, \qquad h_{14} = h_{23} = -\mu G^{1/2}\hat{\gamma}\,, \qquad k_1 = k_3 = 4\mu^2 G \hat{\gamma}^2\,. \quad (2.12)$$

This more compact notation will simplify the expressions throughout the paper. It also helps identifying that the metric and the NS 3-form fall into a particular case of the pp-waves studied in [26]. It was shown there that despite the fact that the crossed $dy^i du$ terms in the metric can in general be brought to du^2 terms via coordinate transformations, the resulting coefficients are u-dependent unless the matrices f_{ij} and $k_{ij} = k_i \delta_{ij}$ commute. From (2.12) it is straightforward to check that this is not so in our case, and we will therefore stick to the form (2.11) where all coefficients are constant.

3. Number of supersymmetries

Let us find the number of supersymmetries that the bosonic type IIB configuration (2.9)-(2.11) preserves. It is possible to show that the total number must be 20 without having to actually solve the Killing spinor equations for the resulting background. The shortcut is based on the observation made it [5] that the Penrose limit and the $\mathrm{SL}(2,\mathbb{R})$ transformation used to deform $AdS_5 \times S^5$ commute. In other words, our background (2.9) can also be obtained[4] as follows:

1. Take the Penrose limit about the $J_{\phi_1} = J_{\phi_2} = J_{\phi_3}$ geodesic of the $AdS_5 \times S^5$ metric (2.1). This leads to the maximally supersymmetric pp-wave in 'magnetic' coordinates[5]

$$\begin{aligned} ds^2_{IIB} &= 2dudv - \mu^2 r^2 du^2 + 4\mu(y^1 dy^2 + y^3 dy^4)du + d\vec{r}^2 + d\vec{y}^2 \,, \\ F_5 &= 4\,\mu\, du \wedge (dr^1 \wedge dr^2 \wedge dr^3 \wedge dr^4 + dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4) \,. \end{aligned} \tag{3.1}$$

2. Compactify y^2 and y^4 on a torus, perform the $\mathrm{SL}(2,\mathbb{R})$ duality along this torus and, finally, decompactify y^2 and y^4.

We can easily compute the explicit form of the 32 Killing spinors for (3.1). Only those which happen to be independent of (y^2, y^4) will survive the $\mathrm{SL}(2,\mathbb{R})$ transformation. The Killing spinor equations that follow from the variation of the IIB gravitino are

$$\left(\partial_M + \frac{1}{4}\omega_M^{mn}\Gamma_{mn} + \frac{i}{5!\;16}(F_5)_{mnpqr}\Gamma^{mnpqr}\Gamma_M\right)\epsilon = 0 \,, \tag{3.2}$$

where $M = 0, \ldots, 9$ is a curved spacetime index, $m, n.. = 0, \ldots, 9$ are tangent space ones, and ϵ is a 10d positive chirality complex Weyl spinor. As explained in appendix A, we will use the indices $\{i, j = 1, \ldots, 4\}$ to denote the y^i coordinates, and indices $\{a, b = 5, \ldots, 8\}$ to denote the r^a ones. We will use the standard pp-wave vielbeins

$$e^v = dv - \mu^2 r^2 du + 2\mu(y^1 dy^2 + y^3 dy^4) \,, \qquad e^u = du \,, \qquad \epsilon^a = dr^a \,, \qquad e^i = dy^i \,, \tag{3.3}$$

which brings the metric to $ds^2 = 2e^u e^v + e^a e^a + e^i e^i$. See appendix A for our index conventions. The non-vanishing components of the spin connection are

$$\omega^{ij} = -f_{ij} du \,, \qquad \omega^{vi} = f_{ij} dy^j \,, \qquad \omega^{va} = -\mu^2 r^a du \,, \tag{3.4}$$

with f_{ij} being the same antisymmetric matrix as in (2.12) but with the deformation parameter set to zero, i.e., $f_{12} = f_{34} = \mu$. Using the reasoning of [27] as a guideline, we rewrite the Killing spinor equation (3.2) as $(\partial_M + i\Omega_M)\epsilon = 0$ with

$$\Omega_v = 0 \,, \tag{3.5}$$

$$\Omega_u = -\frac{1}{2} f_{ij}\Gamma_{ij} - \frac{1}{2}\mu^2 r^a \Gamma_{va} + \frac{\mu}{4}\Gamma_v(\Gamma_{5678} + \Gamma_{1234})\Gamma_u \,, \tag{3.6}$$

$$\Omega_a = \frac{\mu}{4}\Gamma_v \Gamma_{5678}\Gamma_a \,, \tag{3.7}$$

$$\Omega_i = \frac{\mu}{4}\Gamma_v \Gamma_{1234}\Gamma_i + \frac{i}{2} f_{ij}\Gamma_{vj} \,. \tag{3.8}$$

[4]We have checked explicitly that this alternative method leads to the same background.

[5]The name refers to the fact that the light-cone dynamics of a relativistic particle in these coordinates is equivalent to that of a non-relativistic one in a magnetic field.

The first immediate solutions are given by the 16 spinors subject to $\Gamma_v \epsilon = 0$. For these, equations (3.5), (3.7), (3.8) imply that they can only depend on u; such dependence is fixed by (3.6), which becomes a first-order linear ordinary differential equation with constant coefficients, which has a unique solution for each initial value. Because all these 16 spinors are (y^2, y^4)-independent they survive the $\mathrm{SL}(2,\mathbb{R})$ transformation.

Let us now look at the form of the remaining 16 spinors. Nilpotency of Γ_v implies that

$$\Omega_a \Omega_b = 0\,, \qquad \Omega_a \Omega_i = 0\,, \qquad \Omega_i \Omega_j = 0\,, \qquad \forall a, b, i, j\,. \tag{3.9}$$

This implies that both (3.7) and (3.8) are solved by spinors of the form

$$\epsilon(u, r^a, y^i) = \left(1 - i r^a \Omega_a - i y^i \Omega_i\right)\, \chi(u)\,. \tag{3.10}$$

It is straightforward to check that plugging this expression into the remaining equation (3.6) leads again to a first-order linear ordinary differential equation with constant coefficients for $\chi(u)$. The conclusion is that the remaining 16 Killing spinors are all of the form (3.10). Out of them, the only ones that do not depend on (y^2, y^4) are those for which

$$\Omega_2 \chi = 0\,, \qquad \Omega_4 \chi = 0\,. \tag{3.11}$$

Using the explicit values of the Ω's, these conditions can be more properly expressed as

$$\Gamma_{12} \chi = i\chi\,, \qquad \Gamma_{34} \chi = i\chi\,, \tag{3.12}$$

each reducing the dimension of the space of solutions by 1/2. We conclude that only 4 of the 16 spinors (3.10) survive the $\mathrm{SL}(2,\mathbb{R})$ deformation. Physically, these projections select those spinors which are invariant under rotations in the planes (y^1, y^2) and (y^3, y^4).

The conclusion is that the Penrose limit under consideration preserves $16 + 4 = 20$ supersymmetries. The extra 4 ones are often called 'supernumerary'; a very important property [19] is that they are the only ones that become linearly realised in the string worldsheet when the light-cone gauge is imposed, leading to standard properties of supersymmetric field theories like equality of bosonic and fermionic masses.

We end by noting that we have rederived the same results of this section by working out the Killing spinors directly in the final background (2.9). We do not include the computations here, but just point out that, in that case, both projections (3.12) follow directly from the variation of the dilatino.

4. Quantization of the string σ-model

As explained in section 2, the four directions parametrized by $\vec{r}$ are unchanged with respect to the standard maximally supersymmetric pp-wave. We will therefore only consider the four remaining ones, parametrized by y^i, with $i = 1, 2, 3, 4$. We will work in the second coordinate system discussed in section 2; therefore, the configuration is as in (2.9) but with the values of the metric and B-field written in (2.11).

A last remark is that we will not consider the contribution of the vacuum energy as this is guaranteed to cancel among fermions and bosons due to our results of section 3.

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4.1 Quantization of the bosonic sector

We start with the bosonic part of the closed string σ-model,

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(\sqrt{-h}\, h^{\alpha\beta}\, \partial_a X^M \partial_b X^N G_{MN} + \epsilon^{\alpha\beta}\, \partial_a X^M \partial_b X^N B_{MN} \right), \tag{4.1}$$

with $0 \le \sigma \le \pi$. We choose to work with dimensionless worldsheet coordinates, so that the worldsheet lagrangian and hamiltonian are dimensionless too; we will also work in conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$. The equation of motion for the spacetime field u allows us go to the light-cone,

$$u = 2\alpha' p_v\, \tau \equiv \kappa\tau\,, \qquad \kappa = 2\alpha' p_v\,. \tag{4.2}$$

Because of this choice, the spacetime light-cone energy E_{lc} is related to the worldsheet one H_{lc} by

$$E_{lc} = \frac{H_{lc}}{2\alpha' p_v}\,. \tag{4.3}$$

Having fixed the gauge, the equations of motion for the remaining 4 spacetime fields y^i which originated from the S^5_γ are

$$-\ddot{y}^i + (y^i)'' + 2\kappa(f_{ij}\dot{y}^j - h_{ij}y_j') - \kappa^2 k_i y^i = 0\,, \tag{4.4}$$

where, as usual, dots stand for ∂_τ and primes for ∂_σ. To solve these equations, we expand the fields in Fourier modes

$$y^i(\tau,\sigma) = \sum_{n=-\infty}^{\infty} y^i_n(\tau) e^{i2n\sigma}\,, \tag{4.5}$$

which leads to

$$-\ddot{y}^i_n + 2\kappa f_{ij}\dot{y}^j_n - (\kappa^2 k_i + 4n^2) y^i_n - 4in\kappa h_{ij} y^j_n = 0\,. \tag{4.6}$$

Notice that the last is the only term that mixes the physics in the 12-plane with the physics in the 34-plane. The solution to these equations for the particle-like modes ($n = 0$) and the stringy modes $n \neq 0$ are qualitatively very different, and so we treat them separately.

4.1.1 The stringy modes

If $n \neq 0$, we can try to solve the equations by expanding each of the string modes in a standard harmonic oscillator frequency ansatz, $y^i_n(\tau) \sim u^i(\omega_n) e^{iw_n\tau}$. Plugging this expression into (4.6) we obtain the following linear system of equations for the vector $u^i(\omega_n)$,

$$M_{ij}(w_n, n) u^j_n = 0\,, \tag{4.7}$$

with

$$M_{ij}(\omega, n) \equiv (\omega^2 - \kappa^2 k_i - 4n^2)\delta_{ij} + 2i\kappa\omega f_{ij} - 4i\kappa n h_{ij}\,. \tag{4.8}$$

This system admits non-trivial solutions only if the frequencies are such that

$$0 = \det M_{ij}(\omega_n, n) = \left[(\omega_n^2 - 4n^2)^2 - 4\kappa^2\mu^2\omega_n^2\right]^2\,, \tag{4.9}$$

which leads to 4 different admissible frequencies, each with multiplicity 2. Because the determinant does not depend on the sign of ω_n, it therefore suffices to give the explicit expression for the 2 positive different roots, which are

$$\omega_n^{\pm} = \pm\kappa\mu + \sqrt{\kappa^2\mu^2 + 4n^2}\,. \tag{4.10}$$

It is remarkable that, despite the fact that the equations (4.6) and (4.7) depend in a highly non-trivial way on $\hat{\gamma}$, the resulting frequencies are completely independent of it.

Having found the 8 allowed frequencies we now need to determine the corresponding 8 null eigenvectors $u^i(\omega_n)$ satisfying (4.7). It can be checked that all minors of the matrix M_{ij} vanish, which implies that we cannot use the results of [26]; we have to diagonalise the system by brute force. Because each frequency has multiplicity 2, we indeed find a 2d degenerate vector space for any given ω_n, for which a convenient orthogonal basis is provided by the following two vectors in $\mathbb{R}^4$,

$$\begin{aligned} u_1^i(\omega_n) &= (\omega_n^2 - 4n^2\,,\, 2i\omega_n\kappa f\ ,0\,,\, -4inh\kappa)\,,\\ u_2^i(\omega_n) &= (0\,,\, 4inh\kappa\,,\, \omega_n^2 - 4n^2\,,\, 2i\omega_n\kappa f)\,. \end{aligned} \tag{4.11}$$

Summarizing, the most general solution to (4.6) with $n \neq 0$ is a linear combination of the 8 particular solutions described above,

$$\begin{aligned} \frac{y_n^i(\tau)}{\sqrt{2\alpha'}} =& \left[\xi_1^{(n)}\, u_1^i(\omega_n^+) + \xi_2^{(n)}\, u_2^i(\omega_n^+)\right] e^{i\omega_n^+\tau} + \left[\xi_3^{(n)}\, u_1^i(\omega_n^-) + \xi_4^{(n)}\, u_2^i(\omega_n^-)\right] e^{i\omega_n^-\tau} + \\ &+ \left[\xi_5^{(n)} u_1^i(-\omega_n^+) + \xi_6^{(n)} u_2^i(-\omega_n^+)\right] e^{-i\omega_n^+\tau} + \left[\xi_7^{(n)} u_1^i(-\omega_n^-) + \xi_8^{(n)} u_2^i(-\omega_n^-)\right] e^{-i\omega_n^-\tau}\,. \end{aligned}$$

The arbitrary constant coefficients $(\xi_1, \ldots, \xi_8)$ will become operators in the quantum theory. Whereas the first 4 ones are associated to positive frequencies and, so, to left-moving excitations, the last 4 ones are associated to negative frequencies and, so, to right-moving excitations.

Let us now proceed to the quantization of the system. The canonical momenta associated to the dynamical fields $y^i(\tau,\sigma)$ are easily found to be

$$\Pi_i(\tau,\sigma) = \dot{y}^i(\tau,\sigma)\ -\ \kappa f_{ij} y^j(\tau,\sigma)\,. \tag{4.12}$$

Promoting the fields to operators and imposing the equal-time commutators

$$[y^i(\tau,\sigma), \Pi_j(\tau,\sigma')] = i\delta^i_j\delta(\sigma-\sigma')\,, \qquad [y^i(\tau,\sigma), y^j(\tau,\sigma')] = 0 = [\Pi_i(\tau,\sigma), \Pi_j(\tau,\sigma')]\,,$$

leads, after some algebra, to

$$\begin{aligned} \left[\xi_p^{(-n)}, \xi_q^{(n)}\right] &= \epsilon_p\ \delta_{pq}\ (64n^2\kappa^2\mu^2)^{-1} \frac{\omega_n^-}{\omega_n^+(\omega_n^+ - \kappa\mu)}\,, \qquad p,q = 1,2,5,6\\ \left[\xi_p^{(-n)}, \xi_q^{(n)}\right] &= \epsilon_p\ \delta_{pq}\ (64n^2\kappa^2\mu^2)^{-1} \frac{\omega_n^+}{\omega_n^-(\omega_n^+ - \kappa\mu)}\,, \qquad p,q = 3,4,7,8\,, \end{aligned} \tag{4.13}$$

where ϵ_i are signs to be chosen positive for the modes associated to positive frequencies $(p = 1,2,3,4)$ and negative for those associated to negative frequencies $(p = 5,6,7,8)$.

The rescalings needed to define standard normalized creation and annihilation operators $[a_p^{(n)}, a_q^{(n)\dagger}] = \delta_{pq}$ are now obvious. A straightforward, though lengthy, computation shows that the hamiltonian becomes

$$
\begin{aligned}
H_{lc} &= H_0 + \sum_{n>0}^{\infty} H_n\,, \\
H_n &= \omega_n^+ \left(N_1^{(n)} + N_2^{(n)} + N_5^{(n)} + N_6^{(n)}\right) + \omega_n^- \left(N_3^{(n)} + N_4^{(n)} + N_7^{(n)} + N_8^{(n)}\right). \qquad (4.14)
\end{aligned}
$$

Here H_0 is the zero-mode contribution, to be computed below, and $N_p^{(n)} = a_p^{(n)\dagger} a_p^{(n)}$ is the number operator for the corresponding modes. Note that, although generally taken for granted, it was not obvious a priory that the coefficient multiplying each number operator would be precisely its corresponding frequency; this allows us to think of the frequencies of the modes as exactly the quanta of energy needed to create them.

To end up with the quantization of the string modes in the bosonic sector, we need to impose the level-matching condition following from invariance under translations along the worldsheet of the closed string,

$$
\int_0^\pi \Pi_i X'^i = 0\,. \qquad (4.15)
$$

After some algebra this translates into

$$
\sum_{n>0} n \sum_{p=1}^{8} \epsilon_p\, N_p^{(n)} \;=\; 0\,, \qquad (4.16)
$$

which implies

$$
N_1^{(n)} + N_2^{(n)} + N_3^{(n)} + N_4^{(n)} \;=\; N_5^{(n)} + N_6^{(n)} + N_7^{(n)} + N_8^{(n)}\,, \qquad n \neq 0\,. \qquad (4.17)
$$

As in flat space, we need to add as many right-moving excitations to the ground state as left-moving ones.

4.1.2 The particle-like modes

We now study the $n = 0$ modes of the field (4.5). A look at the frequencies (4.10) shows that the extrapolation to $n = 0$ leads to $\omega_0^+ = 2\kappa\mu$ and $\omega_0^- = 0$. For ω_0^+ the story is much like in the $n \neq 0$ case. In particular, the eigenfunctions are obtained by setting $n = 0$ in the expressions (4.11), and they lead to two zero-mode harmonic oscillators.

However, the vanishing of ω_0^- leads one to suspect the appearance of a free particle spectrum. Let us examine this case in detail. We look for the most general solution to the equations of motion (4.6) with $n = 0$. The first remark is that when $n = 0$ the last term in (4.6) vanishes, so that, the dynamics in the y^1y^2-plane decouple from the dynamics in the y^3y^4-plane. Let us concentrate on the first one, for which the equations of motion reduce to

$$
\ddot{y}_n^i - 2\kappa\, f_{ij}\, \dot{y}_n^j \,+\, \kappa^2\, k_i y^i = 0\,, \qquad i = 1, 2, \qquad (4.18)
$$

with

$$f_{12} = -f_{21} = \mu G^{1/2}\,, \qquad (k_1, k_2) = (4\mu^2\hat{\gamma}^2 G, 0)\,. \tag{4.19}$$

Note that if we set the deformation to zero, then $k_i = 0$, so that the system reduces to a Landau problem (i.e. a charged particle moving in a plane threatened by a constant magnetic field). It is well-known that the quantization of this system leads to a ground state with an infinite (but discrete) degeneracy.[6]

As we turn on the deformation, we slightly modify the value of the magnetic field and, more importantly, we introduce a quadratic potential along one of the axis of the plane (in this case, along y^1). The problem is essentially that of a massive charged particle moving in a sheet positively curved along y^1, as depicted in figure 1, and subject to gravity. We would like to point out that our problem is formally identical to that of a particle moving in an anti-Mach metric. See for instance [28, 26]. Indeed, it has been shown in these papers that some pp-wave solutions of type IIB supergravity *with only the NS field turned on* lead to an equation for the zero-modes which is similar to (4.18) but with the sign of the last term is reversed. This leads to a quadratic but repulsive interaction in the Landau plane, i.e., it corresponds to bending the plane of figure 1 in a concave manner. These authors showed that the infinite degeneracy of the Landau ground state is completely broken, and a free-particle spectrum arises. However, because of their negative sign of the quadratic interaction, the kinetic energy of such free-particles appears with the 'wrong' sign in the hamiltonian, leading to a spectrum unbounded from below.

Our case is definitely better behaved. The reason why our supergravity equations are satisfied for positive values of k_i is due to the fact that we have turned on extra fluxes other than the NS 3-form, namely the RR 3- and 5-forms. The ultimate consequence will be that the free-particle spectrum will arise with the standard positive-definite kinetic term.

Let us proceed to the quantization of our problem. The general solution of (4.18) is

$$\begin{aligned} y^1 &= y_0^1 + \sqrt{2\alpha'}\,\Big([\xi_1^{(0)}\,u_1^1(\omega_0^+) + \xi_2^{(0)}\,u_2^1(\omega_0^+)]e^{i\omega_0^+\tau} + \text{c.c.}\Big)\,, \\ y^2 &= y_0^2 + v_2\,\tau + \sqrt{2\alpha'}\,\Big([\xi_1^{(0)}\,u_1^2(\omega_0^+) + \xi_2^{(0)}\,u_2^2(\omega_0^+)]e^{i\omega_0^+\tau} + \text{c.c.}\Big)\,, \end{aligned} \tag{4.20}$$

where $v_2 \equiv 2\kappa\mu\hat{\gamma}^2 G^{1/2} y_0^1$ and, as mentioned above, $\omega_0^+ = 2\kappa\mu$ and the values of the eigenfunctions u_1^i and u_2^i are given by setting $n = 0$ in (4.11). The oscillator part describes the motion of a particle along (non-circular) orbits in the plane. The constant and linear terms describe the position of the center of such orbits in the plane. As we see, the center is allowed to move only along y^2, which is after all an isometry of the plane, and of the metric in the original coordinates (2.9). The velocity v_2 is fixed by (and proportional to) the position in the axis y^1 transverse to the collective motion. Note however that the canonical momenta (π_1, π_2) are non-zero in both directions because, as follows from (4.12),

$$\chi_1 \equiv \pi_1 + \frac{\kappa\mu}{2\pi\alpha'}\,y_0^2 = 0\,, \qquad \chi_2 \equiv \pi_2 - \frac{v_2 + \kappa\mu y_0^1}{2\pi\alpha'}\,. \tag{4.21}$$

Recall however that the canonical momenta π_i are gauge dependent.

[6]See [20] for an analogous situation in a certain Penrose limit of the $AdS_5 \times T^{1,1}$ background and its dual $\mathcal{N} = 1$ SCFT.

Note that the relations (4.21) do not include the velocities, which means that they should be regarded as phase space constraints. Another way of rephrasing it is that $\det \frac{\partial L}{\partial \dot{y}^i \partial \dot{y}^j} = 0$, making it impossible to express the velocities in terms of the momenta. The Poisson bracket of the two constraints is non-vanishing, implying that (χ_1, χ_2) form a system of second-class constraints. The quantization procedure is therefore straightforward: instead of promoting the Poisson bracket to a quantum commutator, one promotes the Dirac bracket, defined for any pair of phase-space functions F, G as

$$\{F, G\}_{\text{D.B.}} = \{F, G\}_{\text{P.B.}} - \{F, \chi_\alpha\}_{\text{P.B.}} C^{\alpha\beta} \{\chi_\beta, G\}_{\text{P.B.}} \,, \tag{4.22}$$

where $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}_{P.B.}$, and $C^{\alpha\beta}$ is its inverse. One of the nicest properties of the Dirac bracket is that the constraints can be imposed either before of after taking the brackets, which implies that we can 'solve' the relations (4.21) once and for all. We choose to solve for the π_i in terms of the y_0^i,

$$\pi_1 = -\frac{\kappa\mu}{2\pi\alpha'}\, y_0^2 \,, \qquad \pi_2 = \frac{v_2 + \kappa\mu y_0^1}{2\pi\alpha'} \,, \tag{4.23}$$

and so we are just left with the pair (y_0^1, y_0^2) subject to the following Dirac bracket,

$$\{y_0^2, y_0^1\}_{D.B.} = \frac{\alpha'}{\kappa\mu}\sqrt{1+\hat{\gamma}^2} = \frac{1}{2\mu p_v}\sqrt{1+\hat{\gamma}^2} \,. \tag{4.24}$$

In other words, the position in the y^1-axis essentially becomes the momentum operator for the position in the y^2-axis. The standard Heinsenberg uncertainty now implies that the it is not possible to resolve the system at distances smaller than the square root of the RHS of (4.24). After an appropriate rescaling, we obtain the standard position-momentum commutation relations,

$$p_{y^2} \equiv \frac{2\mu p_v}{\sqrt{1+\hat{\gamma}^2}}\, y_0^1 \qquad \Rightarrow \qquad \{y_0^2, p_{y^2}\}_{D.B.} = 1 \,. \tag{4.25}$$

Adding the copy of this non-commutative system corresponding to the (y^3, y^4)-plane and the two harmonic oscillators in (4.20), the complete $n = 0$ hamiltonian reads

$$H_0 = \frac{\kappa^2\hat{\gamma}^2}{4\alpha' p_v^2 G}\, \left[(p_{y^2})^2 + (p_{y^4})^2\right] + \omega_0^+ \left[N_1^{(0)} + N_2^{(0)}\right] . \tag{4.26}$$

Let us conclude this section by commenting on the zero deformation limit. An obvious result is that setting $\gamma = 0$ removes the contribution to the energy of the 'free-particle' excitations, leaving behind a highly degenerate ground state. A not so immediate consequence is that this degeneracy becomes discrete as $\gamma \to 0$. The reason is that the behavior of the differential system (4.18) is not smooth in such limit: for $\gamma = 0$ all solutions to (4.18) are of a harmonic oscillator type. So when $\gamma = 0$ we recover the well known infinite but discrete ground state degeneracy of the Landau problem. The non-smoothness of the limit can be understood from the fact that, no matter how small γ is, the extra term in the potential added to the Landau problem radically changes the large y^1 asymptotics.

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4.2 Quantization of the fermionic sector

The generalization of the flat space fermionic GS action to backgrounds with null Killing vectors is [21, 22]

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma\, \partial_\alpha X^M\, \bar{\Theta}(\sqrt{-h}h^{\alpha\beta} - \epsilon^{\alpha\beta}\tau_3)\, \tau_2\, \Gamma_M D_\beta \Theta\,. \tag{4.27}$$

All conventions are explained in the appendix. Here we just remark that Θ is a pair of positive chirality 10d Majorana-Weyl spinors Θ^I $(I = 1, 2)$ which are rotated by the action of the 2×2 Pauli matrices $(\tau_i)_{IJ}$. The derivative D_α is the pull-back to the worldsheet of the type IIB supercovariant derivative.

Because the two 10d spinors Θ^I are Majorana-Weyl, we choose the following adapted basis for the 32×32 gamma matrices

$$\Gamma^M = \begin{pmatrix} 0 & \gamma^M \\ \bar{\gamma}^M & 0 \end{pmatrix}, \qquad \gamma^M \bar{\gamma}^N + \gamma^N \bar{\gamma}^M = 2\eta^{MN}\,, \tag{4.28}$$

where γ^M are the 16×16 matrices (A.2). These γ-matrices should not be confused with $\hat{\gamma}$, the deformation parameter. In this basis, positive chirality MW spinors can be simply written

$$\Theta^I = \begin{pmatrix} \theta^I \\ 0 \end{pmatrix}, \tag{4.29}$$

with θ^I a pair of real 16 component spinor.

Working in the light-cone gauge $\Gamma_v \Theta^I = 0$, or equivalently, $\bar{\gamma}^u \theta^I = 0$, the fermionic lagrangian (4.27) in our background (2.9)-(2.11) receives different types of contributions which we classify according to their origin:

$$\begin{aligned}
(2\pi\alpha')\, L_{\text{kinetic}} &= 2i\kappa \left(\theta^1 \bar{\gamma}^- \partial_+ \theta^1 + \theta^2 \bar{\gamma}^- \partial_- \theta^2\right), \\
(2\pi\alpha') L_{\text{spin-connection}} &= -i\frac{\kappa^2}{4} \left(\theta^1 \bar{\gamma}^- \not{f} \theta^1 + \theta^2 \bar{\gamma}^- \not{f} \theta^2\right), \\
(2\pi\alpha') L_{H_3} &= i\frac{\kappa^2}{4} \left(\theta^1 \bar{\gamma}^- \not{h} \theta^1 - \theta^2 \bar{\gamma}^- \not{h} \theta^2\right), \\
(2\pi\alpha') L_{F_3} &= -2i\kappa^2\, \hat{\gamma}\mu G^{1/2}\; \theta^1 \bar{\gamma}^- \gamma^{13} \theta^2\,, \\
(2\pi\alpha') L_{F_5} &= 2i\mu\kappa^2 G^{1/2}\; \theta^1 \bar{\gamma}^- \gamma^{1234} \theta^2\,,
\end{aligned} \tag{4.30}$$

where we used the standard slash-notation (e.g. $\not{f} = f_{ij}\gamma^{ij}$). Note that whereas the first three terms are chiral, those coming from the RR fields give rise to non-chiral 'mass terms'.

We now want to write down the equations of motion and solve them. It turns out to be convenient to choose a specific representation for the γ-matrices which will allow us to solve the light-cone constraint and reduce the 16×16 system into an 8×8 one. Writing

$$\gamma^i = \begin{pmatrix} 0 & \rho^i \\ (\rho^i)^T & 0 \end{pmatrix}, \qquad i = 1, \ldots, 8\,, \tag{4.31}$$

and using the explicit representation (A.4) for the 8×8 ρ-matrices, we see that the light-cone constraint $\bar{\gamma}^u \theta^I = 0$ is automatically satisfied by spinors of the form

$$\theta^I = \begin{pmatrix} S^I \\ 0 \end{pmatrix}, \tag{4.32}$$

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with S^I a pair of real but otherwise unrestricted 8-component spinors. The equations of motion in terms of such spinors are rather simple-looking

$$
\begin{aligned}
-2\partial_+ S^1 + \frac{\kappa}{4}\left(\not{f} - \not{h}\right) S^1 + \mu G^{1/2}\kappa(\hat{\gamma}\rho^{13} - \rho^{1234})S^2 &= 0\,, \\
-2\partial_- S^2 + \frac{\kappa}{4}\left(\not{f} + \not{h}\right) S^2 + \mu G^{1/2}\kappa(\hat{\gamma}\rho^{13} + \rho^{1234})S^1 &= 0\,,
\end{aligned}
\tag{4.33}
$$

where now slashed tensors imply contraction with the ρ matrices. Expanding in Fourier and frequency modes $S(\tau,\sigma) = \sum_n e^{2in\sigma} e^{i\omega_n \tau}$, these equations become

$$
\begin{pmatrix} -i(\omega_n + 2n) + \frac{\kappa}{4}(\not{f} - \not{h}) & \mu G^{1/2}\kappa(\hat{\gamma}\rho^{13} - \rho^{1234}) \\ \mu G^{1/2}\kappa(\hat{\gamma}\rho^{13} + \rho^{1234}) & -i(\omega_n - 2n) + \frac{\kappa}{4}(\not{f} + \not{h}) \end{pmatrix} \begin{pmatrix} S^1 \\ S^2 \end{pmatrix} = 0\,. \tag{4.34}
$$

We can now express S^2 as an algebraic function of S^1 and solve the resulting equations for S^1. The requirement that the latter admit non-trivial solutions leads to the restriction that the allowed frequencies are precisely the same as in the bosonic sector (4.10). This a consequence of having the supernumerary charges discussed in section 3.

A further straightforward calculation shows that the hamiltonian and the level-matching constraints take exactly the same form as the bosonic ones (4.14) and (4.17), with the replacement of bosonic creation/anihilation operators by fermionic ones.

A worth mentioning subtlety concerns the fermionic zero modes. On the one hand, we find 2 fermionic zero-modes which are harmonic oscillators with $\omega = 2\kappa\mu$: they are superpartners of the corresponding bosonic ones. On the other hand, we find that the superpartners of the free-particle bosonic modes correspond to two spinors constant on the worldsheet which do not contribute to the hamiltonian, as expected from our supersymmetry analysis in section 3.

5. Field theory interpretation and discussion

Let us try to interpret the results of this paper, which were summarized in section 1.1. As far as the 4 directions coming from the AdS_5 is concerned, nothing changes with respect to the maximally supersymmetric pp-wave [18]: exciting their modes corresponds to the addition of spacetime derivatives to the operators. We therefore discuss only the modes coming from the S^5_γ. Indeed, we will only discuss the bosonic modes, as the extension to the fermionic ones is immediate.

We first concentrate on the particle-like ($n = 0$) excitations of the string. We found that there are two $n = 0$ modes which contribute positively and with a continuous spectrum to the energy. Translating (4.26) into field theory variables, we find that they contribute as

$$
\Delta - (J_{\phi_1} + J_{\phi_2} + J_{\phi_3}) = \text{const.}\, \hat{\gamma}^2 \left[(p_{y^2})^2 + (p_{y^4})^2\right]\,, \tag{5.1}
$$

where (p_{y^2}, p_{y^4}) refer to the momenta along the isometric directions of the two modified Landau planes (see figure 1). To make contact between these two directions (y^2, y^4) and the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ along which the $\mathrm{SL}(2,\mathbb{R})$ transformation was performed, recall that the latter corresponded to shifts of (φ_1, φ_2), and that in the Penrose limit we defined (see eq. (2.7))

$$
\varphi_1 = \left(\frac{1}{2G}\right)^{1/2} \frac{-y^2 + \sqrt{3}y^4}{R}\,, \qquad \varphi_2 = -\left(\frac{1}{2G}\right)^{1/2} \frac{y^2 + \sqrt{3}y^4}{R}\,. \tag{5.2}
$$

A first comment is then that (y^2, y^4) parametrise straight lines in the limit $R \to \infty$, leading to a continuous spectrum for their respective momenta. Now, (5.2) induces the following relations:

$$p_{y^2} = \frac{1}{R}\sqrt{\frac{1}{2G}}\,(-J_{\varphi_1} + J_{\varphi_2})\,, \qquad p_{y^4} = -\frac{1}{R}\sqrt{\frac{3}{2G}}\,(J_{\varphi_1} + J_{\varphi_2})\,. \tag{5.3}$$

Recall that, due to (2.3), J_{φ_1} and J_{φ_1} parametrize how far an operator like (1.3) is from having $J_{\Phi_1} = J_{\Phi_2} = J_{\Phi_3}$. Therefore, we conclude that *momentum along the isometric direction of the modified Landau plane corresponds to charge under* $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$*, which in turn corresponds to departures from* $J_{\Phi_1} = J_{\Phi_2} = J_{\Phi_3}$.

This correspondence makes our results fit nicely with some properties which are well-known. On the one hand, we know that only states which are charged under the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ suffer any modification due to the $\mathrm{SL}(2,\mathbb{R})$.[7] The unique bosonic vacuum that we obtain is precisely the single state out of the infinitely many of the original undeformed Landau plane which is $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ invariant. It has $p_{y^2} = p_{y^4} = 0$, and therefore corresponds to the only state with $J_{\Phi_1} = J_{\Phi_2} = J_{\Phi_3}$.

Let us ask what happens to the other states in the undeformed Landau vacuum after the $\mathrm{SL}(2,\mathbb{R})$ duality. At the classical level, we know that if a point-like state has momenta along the torus, then the $\mathrm{SL}(2,\mathbb{R})$ will map it to a state in the deformed background with both momenta and winding along the torus [5]. For certain values of γ, the final state can be interpreted as an extended spinning string similar to those in [30]. Classical extended strings in the γ-deformation of $AdS_5 \times S^5$ were already analyzed by Lunin and Maldacena [5], and it is easy to see that they are all decoupled in our Penrose limit.[8] Nonetheless, some qualitative features can be imported from this classical picture. In particular, the a priori point-like states of the string in our background seem to have a 'minimum length' due to the non-commutative nature of the quantum commutation relations (4.24).

Let us now try to explain why the discrete infinite degeneracy of the Landau problem turns into a *gapless continuous* spectrum. Equation (5.3) tells us that despite the fact that all J_{Φ_i} are of order R^2 in the Penrose limit, their differences must be kept of order R in order to yield finite worldsheet charges p_y, and thus contribute to the light-cone energy. In other words, varying slightly the charges p_y corresponds to roughly exchanging $R \sim N^{1/4}$ times the fields Φ_i in our operators. In the large N limit, the gap between the light-cone energy of operators which differ by a small number of exchanges of the Φ_i operators tends to zero, and the spectrum becomes continuous.

The explicit prediction that we are doing here is that the conformal dimensions of these operators in the deformed theory is given by

$$\Delta - (J_{\phi_1} + J_{\phi_2} + J_{\phi_3}) = g_{YM}^2 N\,\frac{\gamma^2}{3}\,\frac{J_{\varphi_1}^2 + J_{\varphi_2}^2 + J_{\varphi_1}J_{\varphi_2}}{J_{\phi_1} + J_{\phi_2} + J_{\phi_3}}\,, \tag{5.4}$$

[7] See [29] for a recent application of this idea to try to cure some of the unwanted features of the Maldacena-Núñez background.

[8] This is due to the fact that, using the language of section 2, the extended strings of [5] are placed in the S^5_γ in such a way that $\mu_i^2 - \mu_j^2$ remains finite in the limit that we are taking. However, the region that we focus on has $\mu_i^2 - \mu_j^2 \sim 1/R$.

which follows from (4.26), together with (2.10), (4.3) and (5.3), and we recall that the definition of the J_φ's was given in (2.3). Note that, in the Penrose limit, all quantities in the right hand side of (5.4) scale in such a way that the net result is finite.

Having explained the two $n=0$ modes with continuous spectrum, we turn to the two harmonic oscillator ones, which have $\Delta-(J_{\phi_1}+J_{\phi_2}+J_{\phi_3})=2$. The minimal choice is to assume that they correspond to insertions of either $\bar\Phi_1$ or $\Phi_2\bar\Phi_3$, since both have the correct $\Delta-\sum J$, and the first has $p_{y^2}=0$ whereas the second has $p_{y^4}=0$.

Finally, the same reasoning as in the maximally supersymmetric pp-wave [18] works here: the stringy modes correspond to the same replacements/insertions mentioned above with the addition of the corresponding momentum phases. In terms of field theory variables, the contribution of the level n modes to the anomalous dimensions is

$$\Delta-(J_{\phi_1}+J_{\phi_2}+J_{\phi_3})=\pm 1+\sqrt{1+g_{YM}^2 N\,\frac{n^2}{(J_{\phi_1}+J_{\phi_2}+J_{\phi_3})^2}}\,, \tag{5.5}$$

which follows from (4.10), (2.10), and (4.3).

An obvious interesting continuation of this work would be to check these predictions in the field theory. This might be more difficult than in the original BMN case [18, 31] because of the large number of different phases that the deformation introduces to operators with a large number of fields. It would also be nice to see if similar phenomena to those studied here happen to γ-deformations of other SCFTs whose Penrose limits lead to ground states with infinite discrete degeneracy. This is the case, for example, of the Klebanov-Witten theory [32], which has an $AdS_5\times T^{1,1}$ dual admitting a Penrose limit which also leads to a Landau problem [20, 33].

Note added. The same day that this paper was sent to the archive, the paper [34] appeared which has some overlap with sections 2 and 4.1.1.

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A. Conventions

In the background after the Penrose limit we have been consistent in the use of the following notation

curved spacetime coordinates:	X^M	$M,N=u,v,1,2,...8$,
Landau directions:	y^i	$i,j=1,2,3,4$,
unchanged directions:	r^a	$a,b=5,6,7,8$,
worldsheet coordinates:	σ^α	(τ,σ) ,
creation/anihilation modes:	ξ_p	$p,q=1,...,8$.

In the quantization of the fermionic part of the action, we defined the antisymmetric symbol $\epsilon^{\alpha\beta}$ such that $\epsilon^{01} = -1$. The derivative D_α appearing in (4.27) is the pull-back to the worldsheet of the type IIB super covariant derivative,

$$\begin{aligned} D_M &= \nabla_M + \frac{1}{8}\tau_3(H_3)_{MNP}\Gamma^{NP} - \frac{e^\phi}{48}\left[\tau_1(F_3)_{NPQ}\Gamma^{NPQ} + \frac{i}{40}\tau_2(F_5)_{NPQRS}\Gamma^{NPQRS}\right]\Gamma_M\,, \\ \nabla_M &= \partial_M + \frac{1}{4}\omega_M^{NP}\Gamma_{NP}\,. \end{aligned}$$

In the process of reducing the dimensionality of the originally 32-component spinors, we first used a Weyl representation

$$\Gamma^M = \begin{pmatrix} 0 & \gamma^M \\ \bar\gamma^M & 0 \end{pmatrix}, \qquad \gamma^\mu\bar\gamma^\nu + \gamma^\nu\bar\gamma^\mu = 2\eta^{\mu\nu}\,, \tag{A.1}$$

with Γ^M and γ^M being 32×32 and 16×16 respectively, and

$$\gamma^M = (1, \gamma^I, \gamma^9)\,, \qquad \bar\gamma^M = (-1, \gamma^I, \gamma^9)\,, \qquad I = 1, \ldots, 8\,. \tag{A.2}$$

We also used the convenient notation

$$\gamma^{ij} = \frac{1}{2}(\gamma^i\bar\gamma^j - \gamma^j\bar\gamma^i)\,, \qquad \gamma^{ijk} = \frac{1}{6}(\gamma^i\bar\gamma^j\gamma^k + 5\text{ terms})\,, \qquad \text{etc.} \tag{A.3}$$

Choosing the Majorana representation for the Γ-matrices, all γ-matrices are real and symmetric. Note that the 8 matrices γ^I form a representation of SO(8) Clifford algebra. We can use the explicit representation for them in which (see e.g. [35])

$$\gamma^I = \begin{pmatrix} 0 & \rho^I \\ (\rho^I)^T & 0 \end{pmatrix}, \tag{A.4}$$

with

$$\begin{aligned} \rho^1 &= i\tau_2 \times i\tau_2 \times i\tau_2\,, & \rho^2 &= 1 \times \tau_1 \times i\tau_2\,, \\ \rho^3 &= 1 \times \tau_3 \times i\tau_2, & \rho^4 &= \tau_1 \times i\tau_2 \times 1\,, \\ \rho^5 &= \tau_3 \times i\tau_2 \times 1, & \rho^6 &= i\tau_2 \times 1 \times \tau_1, \\ \rho^7 &= i\tau_2 \times 1 \times \tau_3\,, & \rho^8 &= 1 \times 1 \times 1. \end{aligned}$$

The light-cone gauge implies that γ-matrices will only end up acting on spinors of the form (4.32), so we can everywhere replace products of γ-matrices by products of ρ-matrices.

$$\gamma^{I_1}\bar\gamma^{I_2}\ldots \longrightarrow \rho^{I_1}(\rho^{I_2})^T\ldots\,. \tag{A.5}$$

B. The full γ-deformation of $AdS_5 \times S^5$

For the sake of clarity of the exposition, in the paper we avoided writing the full type IIB configuration corresponding to the γ-deformation of $AdS_5 \times S^5$ before the Penrose limit. Adapting to our notation the results of [5] we have,

$$ds^2_\gamma = R^2\left[ds^2_{AdS_5} + \sum_{i=1}^3 \left(d\mu_i^2 + G\mu_i^2 d\phi_i^2\right) + R^4\gamma^2 G\mu_1^2\mu_2^2\mu_3^2\left(\sum_{i=1}^3 d\phi_i\right)^2\right],$$

$$
\begin{aligned}
e^{2\phi} &= e^{2\phi_0} G \,, \\
B_2^{NS} &= \gamma R^4 G \left(\mu_1^2 \mu_1^2 d\phi_1 d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 d\phi_3 + \mu_3^2 \mu_1^2 d\phi_3 d\phi_1 \right) , \\
C_2^{RR} &= -12 \gamma R^4 e^{-\phi_0} \mu_1 \mu_2 \mu_3 \sin\alpha \, d\alpha d\theta d\psi \,, \\
F_5^{RR} &= 4 R^4 e^{-\phi_0} (vol_{AdS_5} + G vol_{S^5}) \,,
\end{aligned}
\tag{B.1}
$$

with

$$
\begin{aligned}
G^{-1} &\equiv 1 + R^4 \gamma^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2) \,, \\
(\mu_1, \mu_2, \mu_3) &= (\cos\alpha, \sin\alpha\cos\theta, \sin\alpha\sin\theta) \,.
\end{aligned}
\tag{B.2}
$$

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